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Homogeneous Finsler spaces of negative curvature☆

Shaoqiang Deng*, Zixin Hou

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, PR China

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Abstract

We prove that a homogeneous Finsler space with non-positive flag curvature and strictly negative Ricci scalar is a simply connected manifold.

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Keywords: Homogeneous Finsler space; Flag curvature; Ricci scalar

Introduction

It is important in Riemann–Finsler geometry to study the relationship between curvature and topology. The Cartan–Hadamard Theorem (cf. [2]) asserts that for a forward geodesically complete connected Finsler space of non-positive flag curvature, the exponential mapping is a C^1 covering projection at every point. This means that the universal covering manifold of this space is C^1 diffeomorphic to Euclidean space. In particular, if this Finsler space is simply connected, then it is C^1 diffeomorphic to Euclidean space. A natural problem is to ask what kind of Finsler spaces of non-positive flag curvature are necessarily simply connected.

In the Riemannian case, there are many excellent works related to this problem. According to Cartan's theory of the symmetric Riemannian manifold [8], a globally symmetric Riemannian space of negative curvature is simply connected. This result was generalized by S. Kobayashi to prove that a homogeneous Riemannian manifold of non-positive sectional curvature and negative definite Ricci tensor is simply connected [9]. Since then, many people have considered the problem of classification of homogeneous Riemannian manifolds of negative curvature (see, for example, [7,13]).

The purpose of this paper is to prove the following

Main Theorem. *Let (M, F) be a connected homogeneous Finsler space of non-positive flag curvature. If the Ricci scalar is everywhere strictly negative, then M is simply connected.*

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* Corresponding author.
E-mail addresses: dengsq@nankai.edu.cn (S. Deng), houzx@nankai.edu.cn (Z. Hou).

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We remark that the Finsler metrics under consideration by us need not be reversible. A Finsler space is called homogeneous if the group of isometries (cf. [4]) acts transitively on the underlying manifold. A homogeneous Finsler space is necessarily (forward and backward) complete. This fact can be proved similarly as in the Riemannian case.

Compared to Kobayashi's proof in the Riemannian case, ours is much more complicated. This is understandable and to be expected. For example, a geodesic in a Finsler space need not be constant speed; the distance function in a Finsler space is in general not symmetric. These facts sometimes cause much difficulty in the Finslerian setting (see the proof of Lemma 3.2 below).

The arrangement of the paper is as follows. In Section 1, we recall some fundamental notions in Finsler geometry such as Chern connection, flag curvature, Ricci scalar. In Section 2, we study Killing vector fields on Finsler manifolds. Some results of this section are needed in the proof of the main theorem. Finally, in Section 3, we complete the proof of the main result.

1. Flag curvature and Ricci scalar

In this section we will recall some basic definitions and notations needed in this paper. In particular, we will introduce the Chern connection and the notions of covariant derivatives, flag curvature, Ricci scalar etc.

Let (M, F) be a Finsler space and $(x^1, x^2, \ldots, x^n)$ be a local coordinate system on an open subset U of M. Then $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ form a basis for the tangent space at any point in U. For $y \in T_x(M)$, $x \in U$, write $y = y^j \frac{\partial}{\partial x^j}$. Then $(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n)$ is a (standard) coordinate system on TU. Using the coefficients g_{ij} and C_{ijk} of the fundamental form and the Cartan tensor (cf. [2] or [3]), we define

$$C^i{}_{jk} = g^{is} C_{sjk},$$

where (g^{ij}) is the inverse matrix of (g_{ij}). The formal Christoffel symbols of the second kind are

$$\gamma^i{}_{jk} = g^{is} \frac{1}{2} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right).$$

They are functions on $TU - \{0\}$. We can also define some other quantities on $TU - \{0\}$ by

$$N^i{}_j(x, y) := \gamma^i{}_{jk} y^k - C^i{}_{jk} \gamma^k{}_{rs} y^r y^s,$$

where $y = y^i \frac{\partial}{\partial x^i} \in T_x(M) - \{0\}$.

The slit tangent bundle $TM - \{0\}$ is a fibre bundle over the manifold M with the natural projection π. Since TM is a vector bundle over M, we have a pull-back bundle $\pi^* TM$ over $TM - \{0\}$. The bundle $\pi^* TM$ admits a unique linear connection, called the Chern connection, which is torsion free and almost g-compatible [1,2]. The coefficients of the connection in the standard coordinate system are

$$\Gamma^l{}_{jk} = \gamma^l{}_{jk} - g^{li} \left(A_{ijs} \frac{N^s{}_k}{F} - A_{jks} \frac{N^s{}_i}{F} + A_{kis} \frac{N^s{}_j}{F} \right).$$

Let $\omega^i_j = \Gamma^i{}_{jk} \mathrm{d}x^k$. To define the flag curvature, we need some differential forms on the manifold $TM - \{0\}$. Let

$$\delta y^i = \mathrm{d}y^i + N^i{}_j \, \mathrm{d}x^j.$$

The curvature 2-forms of the Chern connection are

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k.$$

Since $\Omega^i{}_j$ are 2-forms on the manifold $TM - \{0\}$, they can be expanded as

$$\Omega^i{}_j = \frac{1}{2} R_j{}^i{}_{kl} \mathrm{d}x^k \wedge \mathrm{d}x^l + P_j{}^i{}_{kl} \, \mathrm{d}x^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q_j{}^i{}_{kl} \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}.$$

(it turns out that $Q_j{}^i{}_{kl} = 0$).

Let $\sigma(t)$ be a smooth regular curve in M, with velocity field T. Let $W(t) = W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along σ; define a vector field along σ by

$$\left[\frac{\mathrm{d}W^i}{\mathrm{d}t} + W^j T^k ({\Gamma^i}_{jk})_{(\sigma,T)}\right]\frac{\partial}{\partial x^i}_{|\sigma(t)}.$$

This is called $D_T W$ with reference vector T. We can also define another vector field by

$$\left[\frac{\mathrm{d}W^i}{\mathrm{d}t} + W^j T^k ({\Gamma^i}_{jk})_{(\sigma,W)}\right]\frac{\partial}{\partial x^i}_{|\sigma(t)},$$

which is called $D_T W$ with reference vector W. Moreover, define

$$R(W,T)T := (T^j ({R_j}^i{}_{kl})_{(\sigma,T)} T^l) W^k \frac{\partial}{\partial x^i}.$$

If $\sigma(t)$ is a geodesic and $D_T D_T W$ (with reference vector T) is equal to $-R(W,T)T$, we call W a Jacobi field along $\sigma(t)$.

Now we can define the notion of flag curvature. A flag on M at $x \in M$ is a pair (P, Y), where P is a plane in the tangent space $T_x M$ and Y is a non-zero vector in P. The flag curvature of the flag (P, Y) is defined to be

$$K(P,Y) := \frac{g_Y(R(U,Y)Y,U)}{g_Y(Y,Y)g_Y(U,U) - [g_Y(Y,U)]^2},$$

where $U = U^i\frac{\partial}{\partial x^i}$ is any non-zero vector in P such that $P = \mathrm{span}\{Y, U\}$. It can be shown that the quantity is independent of the selection of U [2]. The Ricci scalar is defined as follows: Let $l = \frac{Y}{F(Y)}$ (the distinguished section). Then select $n-1$ vectors in $T_x(M)$, say $V_1, V_2, \ldots, V_{n-1}$, such that $l, V_1, V_2, \ldots, V_{n-1}$ form an orthonormal basis of $T_x(M)$ with respect to the inner product $g_Y(\cdot,\cdot)$. Let $P_i = \mathrm{span}(Y, V_i)$, $i = 1, 2, \ldots, n-1$. Then the Ricci scalar at Y is defined to be

$$\mathbf{Ric}(Y) = \sum_{i=1}^{n-1} K(P_i, Y).$$

It can be shown that $\mathbf{Ric}(Y)$ is equal to the trace of the endomorphism $U \to R(U,Y)Y$ of the vector space $T_x(M)$ [3].

2. Killing vector fields on Finsler spaces

In this section, we study Killing vector fields on Finsler spaces. Some results will be useful in proving the main theorem. Let (M, F) be a Finsler space, where F is positively homogeneous of degree one (but perhaps not absolutely homogeneous). An isometry of (M, F) is a diffeomorphism σ of M such that $F(d\sigma(Y)) = F(Y)$, $\forall Y \in TM$. It is proved in [4] that a mapping ϕ of M onto itself is an isometry if and only if ϕ is distance preserving, i.e., for any pair of points $x, y \in M$, we have $d(\phi(x), \phi(y)) = d(x, y)$. A vector field X on M is called a Killing vector field if any local one-parameter transformation group φ_t of M generated by X consists of local isometries of M.

In the following we will give a geometric description of Killing vector fields, using Chern's orthonormal frame bundle. Let us first recall the construction of Chern's orthonormal frame bundle of a Finsler space (see [11] for the details). Let $p \in M$. A Chern's orthonormal frame at p is a frame (i.e., a basis of the linear space $T_p(M)$) $\{X_0, X_1, \ldots, X_{n-1}\}$ on $T_p(M)$ such that

(i) $F(X_0) = 1$;
(ii) The vectors $X_0, X_1, \ldots, X_{n-1}$ form an orthonormal basis of $T_p(M)$ with respect to the inner product g_{X_0}, where g is the fundamental form of F.

The set of all Chern's orthonormal frames is denoted by $O_F(M)$ and is called Chern's orthonormal frame bundle of (M, F). $O_F(M)$ is a subbundle of the linear frame bundle $L(M)$ but in general not a principle subbundle of $L(M)$.

The following proposition is a result of [11].

Proposition 1.1. *A diffeomorphism $f : M \to M$ is an isometry of (M, F) if and only if the induced diffeomorphism $\hat{f}$ of f on $L(M)$ keeps Chern's orthonormal frame bundle, i.e., $\hat{f}(O_F(M)) \subset O_F(M)$.*

Now we prove

Proposition 1.2. *A vector field X on a Finsler space (M, F) is a Killing vector field if and only if the natural lift $\hat{X}$ of X to $L(M)$ is tangent to Chern's orthonormal frame bundle $O_F(M)$ at every point of $O_F(M)$.*

Proof. Recall that the natural lift $\hat{X}$ can be obtained in the following way ([10], page 229): for any point $x \in M$, let φ_t be a local one-parameter group of local transformations generated by X in a neighborhood U of x. For each t, φ_t induces a transformation $\hat{\varphi}_t$ of $\pi^{-1}(U)$ onto $\pi^{-1}(\varphi_t(U))$ in a natural manner, where $\pi : L(M) \to M$ is the natural projection. The local one-parameter transformation groups $\{\hat{\varphi}_t\}$ of $L(M)$ obtained in this way induce a vector field on $L(M)$, which is just $\hat{X}$. If X is a Killing vector field, then φ_t are all local isometries of (M, F). By Proposition 1.1, $\hat{\varphi}_t$ maps $O_F(M)$ onto $O_F(M)$. Hence $\hat{X}$ is tangent to $O_F(M)$ at any point of $O_F(M)$. On the other hand, if $\hat{X}$ is tangent to $O_F(M)$ at every point of $O_F(M)$. Then for any $u \in O_F(M)$, the curve $\hat{\varphi}_t(u)$, as an integral curve through u with vector field $\hat{X}$, must be contained in $O_F(M)$. Thus $\hat{\varphi}_t(O_F(M)) \subset O_F(M)$. By Proposition 1.1, φ_t are isometries of (M, F). Hence X is a Killing vector field. □

The proposition implies an important fact that the set of all Killing vector fields of (M, F) forms a Lie algebra. In fact, if X_1, X_2 are Killing vector fields, then the corresponding natural lifts to $L(M)$, $\hat{X}_1, \hat{X}_2$, are tangent to $O_F(M)$ at every point of $O_F(M)$. Since $O_F(M)$ is a submanifold of $L(M)$, $[\hat{X}_1, \hat{X}_2]$ is tangent to $O_F(M)$ at every point of $O_F(M)$. It is obvious that $[\hat{X}_1, \hat{X}_2]$ is the natural lift of $[X_1, X_2]$. Therefore $[X_1, X_2]$ is also a Killing vector field. This proves our assertion. Denote the Lie algebra formed by all Killing vector fields by $\mathfrak{k}(M, F)$ (or simply $\mathfrak{k}(M)$). A natural question is that of determining the dimension of $\mathfrak{k}(M, F)$. In particular, is this Lie algebra of finite dimension? We will give a partial answer to this question in the following. First recall a result of [4] which asserts that the group of isometries of (M, F), denoted by $I(M, F)$ (or simply $I(M)$), is a Lie transformation group of M. We denote the Lie algebra of this group by $\mathfrak{i}(M, F)$ (or $\mathfrak{i}(M)$).

Theorem 1.3. *The Lie algebra $\mathfrak{i}(M, F)$ is isomorphic with the subalgebra of $\mathfrak{k}(M, F)$ consisting of complete Killing vector fields. In particular, if M is compact, then we have $\mathfrak{k}(M, F) \simeq \mathfrak{i}(M, F)$.*

Proof. If $X \in \mathfrak{i}(M, F)$, then $\exp tx$ is a one-parameter group of isometric transformations of (M, F), where exp is the exponential mapping of the Lie group $I(M, F)$. Therefore it induces a Killing vector field on M which is complete. On the other hand, if Y is a complete Killing vector field on M, then it generates a global one-parameter group of isometric transformation of (M, F). Therefore the first assertion follows. If M is compact, then every vector field on M is complete. Therefore the second assertion follows. □

Corollary 1.4. *If (M, F) is a n-dimensional compact Finsler space, then $\dim \mathfrak{k}(M, F) \leq \frac{1}{2}n(n+1)$. Furthermore, if $\dim \mathfrak{k}(M, F) > \frac{1}{2}n(n-1)+1$, then (M, F) is Riemannian of constant curvature.*

Proof. Since the first assertion holds if F is Riemannian, we only need to prove the second one. If $\dim \mathfrak{k}(M, F) > \frac{1}{2}n(n-1)+1$, then by Theorem 1.3, $\dim \mathfrak{i}(M, F) > \frac{1}{2}n(n-1)+1$. Hence $\dim I(M, F) > \frac{1}{2}n(n-1)+1$. Now the assertion follows from a theorem of Wang [12] which asserts that if an n-dimensional Finsler space admits a Lie group of isometries of dimension $> \frac{1}{2}n(n-1)+1$, then it is a Riemannian space of constant curvature. □

It is interesting to consider whether a conclusion similar to Corollary 1.4 holds if M is non-compact. In general, this would be a difficult problem because we do not have a linear connection as in the Riemannian case.

The following results will be useful in the proof of the main result.

Proposition 1.5. *Let (M, F) be a Finsler space and $\sigma(t)$, $a \leq t \leq b$, be a geodesic. Let X be a Killing vector field. Then the restriction of X to σ is a Jacobi field along σ.*

Proof. For any $t \in [a, b]$, we can find a neighborhood U_t of $\sigma(t)$ and a positive number ε_t such that X generates a local one-parameter transformation group of M which is defined on $[-\varepsilon_t, \varepsilon_t] \times U_t$. Since the set $C = \{\sigma(t) | a \leq t \leq b\}$ is compact, we can find a finite number of such open sets U_t whose union covers C. Therefore, we can find a positive number ε such that the local one-parameter group generated by X is defined on $[-\varepsilon, \varepsilon] \times V$, where V is an open subset of M containing C. More precisely, we have a mapping ψ_s of $[-\varepsilon, \varepsilon] \times V$ into M which satisfies the following conditions:

(1) For each $s \in [-\varepsilon, \varepsilon]$, $\psi_t : p \to \psi_t(p)$ is a diffeomorphism of V onto the open set $\psi_s(U)$ of M;
(2) If $s_1, s_2, s_1 + s_2 \in [-\varepsilon, \varepsilon]$, and if $p, \psi_s(p) \in \psi_s(V)$, then

$$\psi_{s_1+s_2}(p) = \psi_{s_1}(\psi_{s_2}(p)).$$

The induced vector field of ψ_s on V is equal to the restriction of X. Therefore ψ_s are local isometries.

Now we define a smooth variation of σ by

$$\sigma(t, s) = \psi_s(\sigma(t)), \quad a \le t \le b, \ -\varepsilon < s < \varepsilon.$$

Then all the t-curves are geodesics. This can be seen from the following observation. Note that σ is locally minimizing, since it is a geodesic [2]. Therefore for any fixed $s \in (-\varepsilon, \varepsilon)$, ψ_s being a local isometry, the curve $\psi_s(t)$, $a \le t \le b$, is locally minimizing. Thus $\psi_s(t)$, $a \le t \le b$, is a geodesic [2]. This proves our assertion. Now by the results of [2] (page 130), the variation vector field of this variation, which is just the restriction of X to σ, is a Jacobi field. □

We also need a result about Killing vector fields on homogeneous Finsler spaces. Let M and N be Finsler spaces and $p : N \to M$ a locally isometric covering projection (cf. [2] for the fundamental properties of covering projections between Finsler spaces). Let G be a connected Lie group of isometries on M which acts transitively on M, $\mathfrak{g} = \text{Lie } G$. Each $X \in \mathfrak{g}$ generates a one-parameter group of isometries of M, and hence can be viewed as a Killing vector field on M. Let X^* be the lift of X to N (by the projection p). The set of all such X^* forms a Lie algebra, denoted by $\mathfrak{g}^*$. Since p is a locally isometric covering projection, we easily see that each X^* is a complete Killing vector field on N. Thus $\mathfrak{g}^*$ is a Lie subalgebra of $\mathfrak{i}(N)$. Let G^* be the (unique) connected Lie subgroup of $I(N)$ corresponding to $\mathfrak{g}^*$. Then we have

Lemma 1.6. *G^* acts transitively on N.*

Proof. The Riemannian case of this result was proved by Wolf in [14] (see also [9]). For the general case, we recall a result in our previous paper [5], which asserts that if a coset space G_1/H_1 of a Lie group G_1 admits a G_1-invariant Finsler metric, then it also admits a G_1-invariant Riemannian metric. Now the Lie group G acts transitively and isometrically on M, so M can be written as G/H, where H is the isotropic subgroup of G at some point. Moreover, the metric on M is invariant under G. Therefore we can find a Riemannian metric g on M which is G-invariant. This means that each $X \in \mathfrak{g}$ can also viewed as a Killing vector field on M with respect to g. Let $g^* = p^*g$; then g^* is a Riemannian metric on N and $p : (N, g^*) \to (M, g)$ is a locally isometric covering projection. By Wolf's result, G^* acts transitively on N. This completes the proof of the lemma. □

3. Proof of the main theorem

Before the proof, we give an example of non-Riemannian homogeneous Finsler space with non-positive flag curvature and strictly negative Ricci scalar.

Example. Consider the Riemannian symmetric pair $(G, K) = (SL(n, \mathbb{R}), SO(n))(n \ge 3)$; the canonical decomposition of the Lie algebra $\mathfrak{g} = \text{Lie } G$ is

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

where $\mathfrak{k} = \text{Lie } K$ consists of all the skew-symmetric matrices in $\mathfrak{g}$ $(=\mathfrak{sl}(n, \mathbb{R}))$ and $\mathfrak{p}$ consists of all the symmetric matrices in $\mathfrak{g}$. Now on $\mathfrak{p}$ we construct a non-Euclidean Minkowski norm F_0 via

$$F_0(X) = \sqrt{Tr(X^2) + \sqrt{Tr(X^4)}} = \sqrt{\sum_{i=1}^{n} \mu_i(X)^2 + \sqrt{\sum_{i=1}^{n} \mu_i(X)^4}}, \quad X \in \mathfrak{p},$$

where $\mu_i(X)$, $1 \le i \le n$, are all the eigenvalues of X. It is easy to check that F_0 is invariant under the adjoint action $Ad(K)$ of K. Therefore we can define a G-invariant Finsler metric F on G/K which is equal to F_0 at the origin if we identify $\mathfrak{p}$ with the tangent space of G/K at the origin [5].

It is a formidable task to compute directly the flag curvature and Ricci scalar of $(G/K, F)$. But we have a simple way to prove that it has non-positive flag curvature and to get the formula of the Ricci scalar. Note that G/K admits an invariant Riemannian metric g such that $(G/K, g)$ is a Riemannian symmetric space which is irreducible of non-compact type. Hence the holonomy group H of g at the origin p_0 $(= eK)$ is equal to $Ad(K)$. Therefore F_0 is invariant under H. Furthermore, the geodesics emanating from p_0 are $\exp tX \cdot p_0$, $X \in \mathfrak{p}$ and the parallel translate of $Y \in \mathfrak{p}$ along $\exp tX \cdot p_0$ is just $(\mathrm{d}\exp tX)_{p_0}(Y)$ ([8], page 208). In particular, F is invariant under the parallel translation along the geodesics of g emanating from the origin. Since g is complete, every point of G/K can be connected to the origin using a geodesic. Now F_0 is invariant under the holonomy group H of the Riemannian metric g at p_0, by Proposition 4.2.2 of [3], F_0 can be extended to a Finsler metric $\bar{F}$ on G/K by parallel translations (of g) such that $\bar{F}$ is affinely equivalent to g. Then by Proposition 4.3.3 of [3], $\bar{F}$ is a Berwald metric (and it is easily seen that the connection of $\bar{F}$ coincides with that of g). Note that $\bar{F}$ and F coincide at p_0 and they are both invariant under the parallel translation of g along the geodesics emanating from the origin p_0. Since every point can be connected to p_0 by a geodesic, we see that $\bar{F}$ and F coincide everywhere. In particular, F is a Berwald metric with the same linear connection of g. Therefore $(G/K, F)$ is a globally symmetric Berwald space of non-compact type and, by Theorem 4.2 of [6], we see that F has non-positive flag curvature. On the other hand, by the formula of the curvature tensor of Riemannian symmetric spaces ([8], page 215) we obtain

$$R(X, Y)Z = -[[X, Y], Z], \quad X, Y, Z \in \mathfrak{p},$$

where R is defined as in Section 1 (for the Finsler metric F). Therefore the Ricci scalar is

$$\mathbf{Ric}(Y) = Tr(-(ad(Y))^2|_{\mathfrak{p}}), \quad Y \in \mathfrak{p},\ Y \neq 0.$$

A direct computation shows that

$$\mathbf{Ric}(Y) = -nTr(Y^2).$$

Since Y is a symmetric matrix, we see that $\mathbf{Ric}(Y) < 0, \forall Y \in \mathfrak{p}, Y \neq 0$. Since $(G/K, F)$ is homogeneous, the conclusions hold at any point of G/K.

Now we turn to the proof of the main theorem. It is just a careful and technical modification of Kobayashi's argument in [9]. We first need to prove two lemmas. Similarly as in the Riemannian case, we call an isometry of a Finsler space a Clifford translation if the distance between a point and its image is the same for every point.

Lemma 3.1. *Let N and M be Finsler spaces and $p : N \to M$ a locally isometric covering projection. If M is homogeneous, then any homeomorphism ϕ of N onto itself satisfying $p \circ \phi = p$ is a Clifford translation of N.*

Proof. Since a distance-preserving mapping of M onto itself is necessarily an isometry [4], we only need to prove that for any two points $y, y' \in N$,

$$d(y', \phi(y')) = d(y, \phi(y')),$$

where d is the distance function of N. Let G, G^* be as in Lemma 1.6. Then G^* is transitive on N. Hence there exists a $\psi \in G^*$ such that $y' = \psi(y)$. Since $p \circ \phi = p$, ϕ induces the identity mapping of M. Hence $\phi_* X^* = X^*, \forall X^* \in \mathfrak{g}^*$. Therefore, in the Lie group G^*, we have

$$\exp(\phi_*(tX^*)) = \exp tX^*, \quad \forall t \in \mathbb{R}.$$

Hence

$$\phi^{-1} \exp(tX^*)\phi = \exp tX^*, \quad \forall t \in \mathbb{R}.$$

From this we conclude that ϕ commutes with every element of the form $\exp tX^*$, $t \in \mathbb{R}$, $X^* \in \mathfrak{g}^*$. Since G^* is connected, it is generated by $\exp U$, where U is a neighborhood of the origin in $\mathfrak{g}^*$. Therefore ϕ commutes with each element of G^*. In particular, we have $\phi \circ \psi = \psi \circ \phi$. Now

$$\begin{aligned} d(y', \phi(y')) &= d(\psi(y), \phi(\psi(y))) \\ &= d(\psi(y), \psi(\phi(y))) \\ &= d(y, \phi(y)). \end{aligned}$$

Thus ϕ is a Clifford translation. □

Lemma 3.2. *Let M, N and ϕ be as in Lemma 3.1. Given $y_0 \in N$, let $y_1 = \phi(y_0)$ and $\gamma^*(t), 0 \le t \le L$, be a unit speed minimizing geodesic from y_0 to y_1. Set $\gamma(t) = p(\gamma^*(t))$. Then $\gamma(t)$ is a unit speed smooth closed geodesic.*

Proof. Since p is a local isometry and γ^* is minimizing, γ is locally minimizing. Therefore γ is a geodesic [2] and it is obviously unit speed. Hence we only need to prove that $\gamma(t)$ is smooth at $\gamma(0) = \gamma(L)$. Suppose that this is not true. Let $\varepsilon > 0$ be so small that the forward metric ball $\mathcal{B}^+_{y_i}(\varepsilon)$, $i = 1, 2$, in N is diffeomorphic (by the mapping p) to the forward metric ball $\mathcal{B}^+_{\gamma(0)}(\varepsilon)$ in M (see [2] for the notation). Let $\delta > 0$ be so small that both $\gamma(L-\delta)$ and $\gamma(\delta)$ are contained in $\mathcal{B}^+_{\gamma(0)}(\varepsilon)$. Then there exists a curve σ in $\mathcal{B}^+_{\gamma(0)}(\varepsilon)$ from $\gamma(L-\delta)$ to $\gamma(\delta)$ with length strictly less than the length of γ from $\gamma(L-\delta)$ to $\gamma(\delta)$. Thus (note that p is a local isometry)

$$\begin{aligned} l(\sigma) &< l(\gamma(t)|_{[L-\delta,\delta]}) \\ &\le l(\gamma(t)_{[L-\delta,L]}) + l(\gamma(t)|_{[0,\delta]}) \\ &= \delta + \delta = 2\delta. \end{aligned}$$

Let σ^* be the curve in $\mathcal{B}^+_{y_1}(\varepsilon)$ such that $p(\sigma^*) = \sigma$ and y^* be the end point of σ^*. Then $y^* = \phi(\gamma^*(\delta))$. Now by the triangular inequality of the distance function in Finsler spaces [2], we have

$$d(\gamma^*(\delta), y^*) \le d(\gamma^*(\delta), \gamma^*(L-\delta)) + d(\gamma^*(L-\delta), y^*).$$

Since γ^* is minimizing, we have

$$d(\gamma^*(\delta), \gamma^*(L-\delta)) = L - 2\delta.$$

On the other hand,

$$d(\gamma^*(L-\delta), y^*) \le l(\sigma^*) = l(\sigma) < 2\delta.$$

Hence

$$\begin{aligned} d(\gamma^*(\delta), \phi(\gamma^*(\delta))) &= d(\gamma^*(\delta), y^*) \\ &< 2L = d(y_0, y_1) \\ &= d(y_0, \phi(y_0)). \end{aligned}$$

Which is a contradiction with Lemma 3.1. □

Proof of the main Theorem. Suppose that M is not simply connected; let N be the universal covering manifold of M and let $p : N \to M$ be the projection. Endow N with the Finsler metric F^* defined by

$$F^*(y) = F(dp(y)), \quad y \in TN.$$

Then $p : N \to M$ is a locally isometric covering projection. Since M is not simply connected, p is not a diffeomorphism. This means that there exists a non-trivial (i.e., not equal to the identity transformation) homeomorphism ϕ of N such that $p \circ \phi = p$. By Lemmas 3.1 and 3.2, we can find a closed smooth unit speed geodesic on M, say $\gamma(t), 0 \le t \le L$, where $L > 0$. Let T be the tangent vector field of γ and V be any Killing vector field on M. Denote $T_t = T|_{\gamma(t)}$. Define a non-negative function $f(t)$, $-\infty < t < \infty$, as follows:

$$f(t) = g_{T_t}(V, V), \quad \text{for } 0 \le t \le L,$$

and then extend it to a periodic function of period L. Note that since $\gamma(t)$ is unit speed, T_t is everywhere non-zero. Thus $f(t)$ is smooth on $(0, L)$ (cf. [2]). By Lemma 3.2, $f(t)$ is smooth for all t.

Let

$$V' = D_T V, \quad V'' = D_T V',$$

where the covariant derivatives are taken with reference vector T. By Proposition 1.5, V is a Jacobi field along γ, i.e., $V'' = -R(V, T)T$. Therefore we have ([2], page 136)

$$\begin{aligned} f'(t) &= 2g_{T_t}(V, V'), \\ f''(t) &= 2g_{T_t}(V', V') - 2g_{T_t}(R(V, T)T, V). \end{aligned}$$

Since (M, F) has non-positive flag curvature, we have $g_{T_t}(R(V, T)T, V) \leq 0, \forall t$. Thus $f''(t) \geq 0$. Since f is a periodic smooth function, this implies that $f(t)$ is a constant function. Thus $f''(t) = 0$. Therefore $g_{T_t}(V', V') = 0$ and $g_{T_t}(R(V, T)T, V) = 0$, for any t.

Now M is a homogeneous Finsler space. Therefore we can write $M = G/H$, where G is a connected Lie group of isometries which is transitive on M and H is the isotropic subgroup of G at $\gamma(0)$. As we mentioned before, there exists a G-invariant Riemannian metric on G/H. In particular, G/H is a reductive homogeneous manifold. Hence there exits a subspace $\mathfrak{m}$ of $\mathfrak{g}$ satisfying $Ad(h)(\mathfrak{m}) \subset \mathfrak{m},\ h \in H$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{(direct sum of subspaces)}.$$

There is a canonical way to identify $\mathfrak{m}$ with the tangent space $T_{eH}(G/H) = T_{\gamma(0)}(M)$. For any $X \in \mathfrak{m}$, the one-parameter subgroup $\exp tX$ induces a Killing vector field on $M = G/H$ which is equal to X at $\gamma(0) = eH$. Hence $g_{T_0}(R(X, T)T, X) = 0, \forall X \in \mathfrak{m}$. But this is a contradiction with the assumption that $\mathbf{Ric}(T_0) > 0$. The contradiction comes from the assumption that M is not simply connected. □

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